Lie derivative

(idea) by chomps 2 **C!**s Wed May 17 2006 at 22:26:36

A lie derivative (pronounced "lee", named after the mathematician Sophus Lie) is a well-defined way of taking derivatives of vectors on a manifold. Computationally, it can be quite simple, though conceptually it's actually very subtle. The depth of this concept requires a good understanding of the tangent space at different points on a manifold.

Given a differential manifold M, it is useful to be able to take derivatives of vector fields. However, until we provide additional information, the concept of a "derivative" of a vector field is not well-defined.

Comparing Tangent Vectors

We want to think of the derivative as "the rate at which a vector field changes as we move a given distance" in the manifold. A small problem with this is that there is as yet no notion of "distance" in our manifold. This is additional structure that we would need to impose, if we wanted to include "distance" in our definition of the derivative. A much greater fundamental problem is that we are trying to measure the rate of change of a vector field as we move from one point to another in the manifold, which means we are implicitly comparing tangent vectors defined at different points p and q in M. There is no god-given way to do this, since tangent vectors at p live in the tangent space T_pM, and tangent vectors at q live in a different space, T_qM.

There are many ways of defining maps between these two spaces, but there is no special or natural map. Choosing a particular map between tangent spaces on the manifold imposes additional structure, and this structure is known as a connection. A derivative which uses a connection defined on M is called a covariant derivative, and will not be used here.

The Lie Derivative provides an alternative method for differentiating vector fields, which does not require a connection. Instead, the additional information specified to compare tangent vectors is known as a congruence of curves.

Congruence of Curves

A congruence of curves defined on a manifold M is simply a smoothly varying family of curves which fill$^+$ the manifold, without intersecting. Each point p in the manifold lies in exactly one curve.

Here is the key concept which allows the lie derivative to function computationally: As was stated in the tangent space node, each vector in T_pM can be considered the velocity of a curve passing through p. In other words, T_pM is the space of equivalence classes of curves passing through p. Thus, any given curve defines a vector at each point in the curve. Therefore, a congruence of curves defines a *smooth vector field* at each point in the manifold.

Can we go the other direction? That is, given a smoothly varying vector field **W** on M, can we produce a unique congruence of curves in M, such that each point p in M is associated with a curve whose velocity at p is equal to $\mathbf{W_p}$, the vector field evaluated at p? The short answer is yes,++ and the resulting congruence is called the set of integral curves of **W**.

So, let us take our vector field **W** which we want to differentiate, and transform it into a congruence of curves using this method. As we know, **W** is also associated with a directional derivative operator at each point, which we shall call $\partial/\partial\mu$. The integral curves of **W** can then be parameterized by the parameter μ. We can write an integral curve of **W** passing through p as $\alpha_p(\mu)$. Now, as was stated previously, we need to provide an additional congruence of curves in order to differentiate **W** properly. Equivalently, we can provide a vector field **V**, since we know **V** is itself uniquely associated with some congruence of curves. We shall write **V** as $\partial/\partial\lambda$, and the integral curves of **V** will be parameterized by λ.

So, we have two congruences of curves. How do we take the derivative of one with respect to another? Conceptually, we want to look at how the curves of **W** change when we move a small distance $\Delta\lambda$ along curves of **V**. We still have to deal with the issue of comparing vectors at different points in the manifold; we've just transformed the problem into comparing *curves* at different points in the manifold. Fortunately, the congruence of curves given by **V** gives us a natural way of transporting a curve of **W** to different points on the manifold. We can define a *new* congruence of curves about the point p in the following manner:

Lie Dragging (You may need to draw a picture to follow along)

At p, look at the integral curve of **W** passing through p. Call this curve $\alpha_\lambda(\mu)$. Move a distance $\Delta\lambda$ along the integral curve of **V** passing through p. Call this new point q. To produce a transported curve $\alpha^*_{\lambda + \Delta\lambda}(\mu)$ passing through q, simply transport each point in $\alpha_\lambda(\mu)$ this same distance $\Delta\lambda$ along the integral curve passing through $\alpha_\lambda(\mu)$. This produces a new curve which we can compare with $\alpha_{\lambda + \Delta\lambda}(\mu)$ by simply taking the difference between their velocities:

$$\Delta_V [W] = \alpha'(\mu)|_{\lambda + \Delta\lambda} - (\alpha^*_{\lambda + \Delta\lambda}(\mu))'$$

A note about notation: we've introduced a great many concepts at once, and it's good to keep our head on straight about why things are written down the way they are. $\alpha_\lambda(\mu)$ is an integral curve of **W** passing through p with parameter μ. We write the subscript "λ" instead of "p" to accentuate the fact that p is given as a point on an integral curve of **V**, parameterized by λ. $\alpha_{\lambda + \Delta\lambda}(\mu)$ is simply another integral curve of **W**, this one instead passing through q, the point gotten by moving a distance $\Delta\lambda$ along an integral curve of **V**. $\alpha^*_{\lambda + \Delta\lambda}(\mu)$ is *not* an integral curve. It is the curve found by transporting $\alpha_\lambda(\mu)$ a distance $\Delta\lambda$ along integral curves of **V** passing through each μ of $\alpha_\lambda(\mu)$. The family of curves produced in this manner is said to be lie dragged. $\alpha_{\lambda + \Delta\lambda}(\mu)$ and $\alpha^*_{\lambda + \Delta\lambda}(\mu)$ intersect each other at q, which is why we can compare their velocities.

Now, why did I write the velocities of α and α^* in the way I did? Well, since $\alpha_{\lambda + \Delta\lambda}(\mu)$ is an integral curve of **W**, the velocity of this curve is exactly what we mean by $\mathbf{W}_q$, the vector field evaluated at q. This can be written by just taking the velocity of integral curves at arbitrary λ, and evaluating it specifically at the point q, which corresponds to $\lambda + \Delta\lambda$ (I'm consistently using the convention that a prime means differentiation by μ). For α^*, we must first transport the curve *before* computing its velocity. This is noted symbolically by putting the subscript $\lambda + \Delta\lambda$ *inside* the parentheses. This will become important shortly.

Computation of the Lie Derivative

We can turn this difference into a derivative by dividing by $\Delta\lambda$ and taking the limit as $\Delta\lambda$ goes to zero. This specifies the lie derivative:

$$\pounds_V[W] = \mathrm{Lim}_{(\Delta\lambda \to 0)}\, [\alpha'(\mu)|_{\lambda + \Delta\lambda} - (\alpha^*_{\lambda + \Delta\lambda}(\mu))'] / \Delta\lambda$$

The simplest way to compute this is to expand these terms to first order in a taylor series in $\Delta\lambda$ (higher order terms vanish when taking the limit $\Delta\lambda \to 0$):

$$\pounds_V[W] = \mathrm{Lim}_{(\Delta\lambda \to 0)}\, [\alpha'(\mu)|_{\lambda} + \Delta\lambda(\partial/\partial\lambda)\alpha'(\mu)_{\lambda} - (\alpha_{\lambda}(\mu) + \Delta\lambda(\partial/\partial\lambda)\alpha_{\lambda}(\mu))'] / \Delta\lambda$$

$$= \mathrm{Lim}_{(\Delta\lambda \to 0)}\, [\Delta\lambda(\partial/\partial\lambda)\alpha'(\mu)_{\lambda} - (\Delta\lambda(\partial/\partial\lambda)\alpha_{\lambda}(\mu))'] / \Delta\lambda$$

The $\Delta\lambda$'s cancel, so we can just get rid of the limit:

$$= (\partial/\partial\lambda)(\partial/\partial\mu)\, \alpha_{\lambda}(\mu) - (\partial/\partial\mu)(\partial/\partial\lambda)\, \alpha_{\lambda}(\mu)$$

$$= [(\partial/\partial\lambda)(\partial/\partial\mu) - (\partial/\partial\mu)(\partial/\partial\lambda)]\, \alpha_{\lambda}(\mu)$$

In a particular coordinate system:

$$= V^j\, \partial/\partial x^j\, [W^i\, \partial/\partial x^i\, (\alpha_{\lambda}(\mu))] - W^j\, \partial/\partial x^j\, [V^i\, \partial/\partial x^i\, (\alpha_{\lambda}(\mu))]$$

The second derivatives of α cancel, and we get:

$$\pounds_V[W] = [V^j\, \partial W^i/\partial x^j - W^j\, \partial V^i/\partial x^j]\, \partial\alpha_{\lambda}(\mu)/\partial x^i$$

We've been interpreting this as the velocity of a curve, α, but we can now think of this as a directional derivative operator acting on the function $\alpha_{\lambda}(\mu)$. In this way, it is readily seen that the components of the vector we've produced are simply the coefficients of $\partial\alpha/\partial x^i$:

$$\pounds_V[W]^i = V^j\, \partial W^i/\partial x^j - W^j\, \partial V^i/\partial x^j$$

We usually write this as the commutator, [**V**,**W**], meaning the result of commuting directional derivative operators **V** and **W**. Written this way, it is often simply called the lie bracket of **V** with **W**.

Lie Derivatives of Other Tensors

We don't have to stop here; we can now take the Lie derivative of arbitrary tensors. For example, we can take the lie derivative of a one-form. We first must fix our lie derivative with two reasonable requirements. First, the lie derivative of a scalar is just the directional derivative:

$£_V[f] = \partial f/\partial\lambda$

Then we note that a scalar function can be formed by operating with a one-form on a vector field:

$\boldsymbol{\omega}(\mathbf{W}) = \omega_i W^i$

Then we finally require that our derivative satisfies a Leibnitz rule,

$£_V[\omega_i W^i] = £_V[\omega]_i W^i + \omega_i £_V[W]^i$

We can then compute all of these terms, plugging in a coordinate basis vector field for $\mathbf{W} = \boldsymbol{\partial}_\mathbf{j}$ ($W^i = \delta^i{}_j$):

$\partial(\omega_j)/\partial\lambda = £_V[\omega]_j + \omega_i £_V[\partial_j]^i$

$V^i \partial(\omega_j)/\partial x^i = £_V[\omega]_j + \omega_i (-\partial_j V^i)$

$£_V[\omega]_j = V^i \partial(\omega_j)/\partial x^i + \omega_i \partial V^i/\partial x^j$

In a similar fashion, we can compute the lie derivative of tensors of arbitrary rank. Generally, the lie derivative is most useful in its rank-1 interpretation, the change in the congruence of curves as described above. In this case, it is also simpler computationally, as it is just given by the lie bracket [**V**,**W**].

Coordinate Bases

Now that we have a new way of comparing tangent spaces, how can we make use of it? Well, the most common use appears when we look at basis vectors that we want to use in different tangent spaces. An important question is, when we choose a set of basis vectors at each tangent space, and this set of basis vectors varies smoothly in the manifold, can we find a coordinate chart in some neighborhood of a point p whose coordinate basis vectors correspond to our choice of basis vectors? In other words, given a set of basis vectors $\{\mathbf{e}_\mu(p)\}$, can we find a coordinate system $\{x^\mu\}$ whose partial derivatives $\{\partial/\partial x^\mu\}$ are the associated directional derivative operators of $\{\mathbf{e}_\mu\}$?

Before telling you why, let me just tell you the answer. The necessary and sufficient conditions for this to be possible is that the lie brackets of all the basis vectors vanish: $[\mathbf{e}_\mu(p), \mathbf{e}_\nu(p)] = 0$, for all μ, ν.

Computationally, it's easy to see why this is a necessary condition. If it's possible to write $\{\mathbf{e}_\mu\}$ as a set of partial derivatives $\{\partial_\mu\}$, then:

$[\mathbf{e}_\mu, \mathbf{e}_\nu] = [\partial_\mu, \partial_\nu] = 0$, because partial derivatives commute (when acting on smooth functions). So, if the lie bracket does not vanish, clearly we cannot write the vectors in terms of partial derivatives of a given coordinate system. However, if the lie bracket *does* vanish, how do we know we can always find such an appropriate coordinate system?

I don't intend to give a formal proof, but such a coordinate system can always be found via the integral curves of $\{\mathbf{e}_\mu(p)\}$. The reason this works (and fails when the lie bracket does not vanish) is that the integral curves agree when we drag them in the way we did before, when calculating $£_V[W]$. Since the lie derivative is zero, that means that $\alpha(\mu)|_{\lambda + \Delta\lambda} = \alpha^*_{\lambda + \Delta\lambda}(\mu)$, i.e. the lie-dragged curve is the same as the integral curve. This insures that our coordinate system is not ambiguous (when we move a parameter distance $\Delta\mu$ along one coordinate then a distance $\Delta\lambda$ along another, we get the same result as if we reverse the order). There are deep topological reasons behind all of this, but this is the basic idea.

How the Lie Derivative differs from the Covariant Derivative

If we are given a vector field **V**, we specify the lie derivative, $£_V$. If we are given a connection Γ, we specify the covariant derivative, ∇_μ. You might now be tempted to ask, is there a relationship between **V** and Γ? That is, given a vector field, **V**, can we produce a connection, Γ, such that $£_V = \nabla$?

The simple answer is no. The lie derivative and the covariant derivative are simply two different beasts. One way of understanding this is to note that a lie derivative is a map from (p,q) tensors to (p,q) tensors, and the covariant derivative is a map from (p,q) tensors to (p,q+1) tensors. The "equation" $£_V = \nabla$ simply makes no sense. It is possible to write down some relationships between the two, but it is really best to think of them as different objects which live in different spaces.

+Really, we're just looking at some small region of M. Thus, the lines of longitude on a sphere work, if we stay away from the poles. If we want to work near the poles, we can use some other congruence of curves. You're not always able to find a congruence that works globally, but you can always find one locally, which is good enough for us.

++The long answer involves a proof, and perhaps some way of constructing these curves. This probably calls for a writeup on the exponential map.

Thanks to unperson for providing some ideas on how to clarify this writeup.

http://everything2.com/?node=Lie+derivative http://everything2.com/?node_id=1807527